Date :

Collatz conjecture - Veracity or Fallacy...... ?

Abstract : Collatz problem is one the simplest complicated problem of number theory. It says that a seemingly simple program of two steps terminates for all no. s.

Take any, natural no. m>o

n : = m

repeat

if (n is odd) then n : = 3n+1,

else n : = n/2

untill (n=1)

Conjecture1. For all positive integers m, the program above terminates.

As no.s are infinite it may be untrue. For a counter example it is a must that in the series any one no. repeats so it may be cyclic from n to n. Whether or not it is true is the post discussed in following pages.

Meaning of Symobls:

x^K - x raised to the power of 'k'

= - is equal to

$\cong$ - is equivalent to

> - is greater than

< - is less than

$\Rightarrow$ - implies

$\therefore$ - Therefore

$\because$ - because

Main body : Collatz problem is one of the most pinching problems of modern mathematics owing to the fact that it is so simple and yet so tough that it resists all solutions. Also known as '3n+1' problem it is a simple algorithm of two simpler steps and asks for whether these two continue for even or not.

To be more precise collatz problem states that the programme of following two steps terminates for all no.s.

Take any, natural no. m>o

n : = m

repeat

if (n is odd) then n : = 3xn+1,

else n : = n/2

untill (n=1)

Conjecture1. For all positive integers m, the program above terminates.

It just means if n is even half it and if it is odd mltiply it by three and add one and go on till n=1 i e smallest natural no.

The complication with this problem is that if it ends for all no. s then how to establish the very fact?

Their is infact one way which uses principle of mathematical induction in multiple steps which resolves the problem as true. Hence collatz conjecture is a veracity.

Before proving it let us work out certain basics.

It starts with the fact that for all even integers it holds if it holds for all odd integers. It is due to the fact that eventually all even 2n will become 2n/n (By step II)

= n

Secondly all odd integers are either 4n+1 or 4n-1. It would be established that odd intergers of the form 4n-1 reduce to 4n+1 after multiple application of step (1) & (II), with two exceptions off course.

We would hereforth write the steps in the form of tree where each branch will follow the steps (1) & (II).

A example follows:-

4n - 1	(odd)
12n - 2	[3n+1]
6n - 1	[n/2]
18n - 2	[3n+1]
9n - 1	[Two branches for]

18n+8   18n - 1   n = even

n = odd

9n + 4   54n - 2   [3n + 1]

27n - 1   [n/2]

18n + 4   18n + 13   etc.

9n + 2   54n + 40

18n + 2   18n + 1   27n + 20

9n + 1

Now we take four cases ie n = 4k , 4k ± 1 or 4k + 2.

$\therefore$ k = 4k + 1   says

4n - 1

4 (4k + 1) - 1   [odd : 3n + 1]

12 (4k + 1) - 2   [Even : n/2]

6 (4k + 1) - 1

6.4 k + 5 $\cong$ 4 k + 1

k =   4k + 2   says

4n - 1

4 (4k + 2) - 1   [odd : 3n + 1]

12 (4k + 1) - 2   [Even : n/2]

6 (4k + 2) - 1   [odd : 3n + 1]

18 (4k + 2) - 2   [even : n/2]

9 (4k + 2) - 1

4.9 k + 17 $\cong$ 4k + 1

k=   4k says

4.4k - 1   [odd : 3n + 1]

12.4k - 2   [even : n/2]

6.4k -1   [odd : 3n + 1]

18.4k - 1   [even : n/2]

9.4k - 1   [odd : 3n + 1]

108k - 2   [even : n/2]

54k - 1

54 (2k + 1) - 1 108k - 1 [one even and other odd]

108k + 53 $\cong$ 4 k + 1

Now 54 $\cong$ 4k + 2

Any no. of the form (4n + 2) k - 1

reduces to 4n + 1 if k = odd & if k= even then after applying rule (II)

it reduces to

3.2 (4n + 2). k - 3 + 1 [Even : n/2]

or 3.(4n + 2) . k - 1

$\therefore$ if k = odd it reduces to 4k + 1 form

$\therefore$ if k = even = 2^n then it is of the form 4k - 1 otherwise not

Since each 3n + 1 raises one power of 3 & $3^n.2 - 1 = (2 + 1)^n. 2 - 1$

$\cong$ 4k + 1 form.

$\therefore$ only 54.2n - 1 is remaining form of 4.4k - 1 which does not reduces to 4k + 1 form.

$\therefore$ Apart from proving all the no. s of the form 4n + 1 we got to prove it for 54.2n - 1 also .... (A)

It is the first exception.

lastly n = 4k - 1 says

4(4k - 1) - 1

12 (4k - 1) - 2 [Even : n/2]

6 (4k - 1) - 1 [odd : 3n + 1]

18 (4k - 1) - 2 [Even : n/2]

9 (4k - 1) - 1

9.4k - 10

9.2k - 10

9.2k - 5

18k - 5 $\cong$ 18k + 18 - 5

$\cong \quad 18k + 13$ .... (B)

It is the second exception.

$\therefore$ 18k + 13 should also be encountered

apart from all the no. s of the form 4n + 1.

$\therefore$ If collatz conjecture holds for all the no. s of the form

4n + 1 .... (I)

$54.2^n - 1$................................(A)

18n + 13 ....(B)

It is true.

We would start with .... (A)

Now $54.2^n - 1 \quad \cong \quad 27.2n - 1 \cong 3^3 . 2^n - 1$

We can easily see that by repeated application of steps (1) & (2) exponent of 3 will rise and that of 2 will fall.

$\therefore$ The end result would be $3^k - 1$

$\therefore$ We have to establish that it holds for all no. s of the form$3^k - 1$

Now $3^k - 1 \cong 3^n - 1$

3n - 1

9n - 2

18n - 2   18n + 7 $\cong$ 3n + 1

9n - 1

9n - 1

27n - 2 $\quad \cong \quad$ 3n + 1

$\therefore$ n = 3k - 1 is true if n = 3n + 1 is true

Now each 3n + 1 can be derived from a subseries of 4n + 1.

That is 4n + 1

12n + 4

6n + 2

3n + 1

$\Rightarrow$ Truth of 4n + 1 implies the truth of 3n + 1

IIIr results for 18n + 13 ie .... (B)

∴ Eventually exceptions (A) & (B) also find themselvelves burried in the truth of $n = 4k + 1$.

∴ their is only one case $n = 4k + 1$ but that isn't that small either.

Till now we have proved that collatz conjecture holds if it holds for all $n = 4k + 1$ type integers. But that isn't half the battle won because one such exception could lead the whole result to a fallacy.

Now comes the vital part of the proof ie. the final case $n = 4k + 1$

The grand finale................,

The essence of using mathematical induction lies in the fact that if every no. of the form $4k + 1$ terminates and implies that $4k + 5$ also terminates then its true.

The proof is divided in two parts. The first part is the first step of two step induction. Through induction it is established that collatz conjecture holds for all no. s of the $4k + 1$ form except for $3^2. 2^n +1$

In Second step it is established that it holds for all no. s of the form $3^2.2^n + 1$

Starting with the basic step ie $k = 1$

$n = 4.1 + 1 =$ 5
|
16
|
8
|
4
|
2
|
1

Now assuming it to be true for $4k + 1$.

lets prove it to be true for $4k + 5$.

The proof assumes that it a no. in the tree of $4k + 1$ appears in the tree of $4k +5$ the that part of the tree is assumed to be true. To make it clear that if in the series of 4.2 +n 1 ie 9 any of the 16, 8, 4, 2 occurs then it is true at that step only.

We first make our reference tree for $4n + 1$

(I) $4n + 1$

$12n + 4$

$6n + 2$

$3n + 1$

$6n + 1$ $6n + 4$

$18n + 4$ $3n + 2$

$9n + 2$

$18n + 2$ $18n + 11$

$9n + 1$

$18n + 1$ $18n + 10$

$36n + 1$ $36n + 19$ $9n + 5$

Next comes our tree for $4k + 5$.

(II) $4kn + 5 = 4n + 5$

$12n + 16$

$6n + 8$

$3n + 4$

$6n + 7$ $6n + 4$

$18n + 22$

$9n + 11$

$18n + 11$ $18n + 20$

$9n + 10$

$18n + 10$ $18n + 19$

Comparing the two trees we get that

in (II) the series end at $6n + 4$ as it appears is (I) also.

IIIrd results for $18n + 11$ and $18n + 10$.

$\therefore$ only no. left out is $18n + 19$

ie $18(n + 1) + 1$

Now whenever n = even it comes in the series of $4n + 1$ as n = odd. Hence $n + 1$ = even or by repeated application $n + 1 \cong 2^k$.

$\therefore$ Only values to be settled are of the form $3^2 . 2^n + 1$

Hence collatz conjecture is a veracity if it holds for all the no. s of the form $9.2^n + 1$.

Now we would again establish this result through inducation.

Hence we will prove it to be true for P (n) = $9.2^n + 1$, offcourse through induction.

For n = 1 P (1) = 19 $\cong$ 58 $\cong$ 29 $\cong$ 88

$\cong$ 44 $\cong$ 22 $\cong$ 11 $\cong$ 34

$\cong$ 17 $\cong$ 52 $\cong$ 26 $\cong$ 13 $\cong$ 40

$\cong$ 20 $\cong$ 10 $\cong$ 5 $\cong$ 16

$\cong$ 8 $\cong$ 4 $\cong$ 2 $\cong$ 1

Now let it hold for

$P(n) = 9.2^n + 1$

$\Rightarrow$ Every no. of the form less than

$9.2^n + 1$ and of the form 4n + 1

Satisfies the collatz conjecture as proved before. (in step one of induction).

$\therefore$ $P(n + 1) = 9.2^{n+1} + 1$

$\cong 27.2^{n+1} + 4 \cong 27.2^n + 2$

$\cong 27.2^{n-1} + 1$

$\Rightarrow$ In one step no. is approximately increased 3 times and decreased 4 times

$\therefore$ Net decreased ins around 3/4 in each step. Hence in three steps it would be 27/64 < 1/2 $\Rightarrow$ New no. is less than 1/2 of $9.2^{n+1}+1$ and of the form 4k + 1.

$\Rightarrow$ In tree steps we would get a no. less than 1/2 $[9.2^{n+1} + 1]$

ie. 9.2n + 1.

Hence it holds for $9.2^{n+1} + 1$

Since it holds for each no. less than $9.2^n + 1$

For three steps to happen the min.

value of 'n' is 8 since the no. has to be reduced $1/2^6$ times and the resulting no. should be of the form 4k + 1.

$\therefore$ We have to manually prove it for all the values of n = 1,2, ...7,

For n= 1 it has been done.

n= 2, 9.4 = 37 $\cong$ 112 $\cong$ 56 $\cong$ 28

$\cong$ 14 $\cong$ 7 $\cong$ 22 $\cong$ 11 $\cong$ 34

$\cong$ 17 $\cong$ 52 $\cong$ 26 $\cong$ 13

$\cong$ 40 $\cong$ 20 $\cong$ 10 $\cong$ 5 $\cong$ 16

$\cong$ 8 $\cong$ 4 $\cong$ 2 $\cong$ 1

n = 3 9.8 + 1 = 73 $\cong$ 220 $\cong$ 110 $\cong$ 55

$\cong$ 166 $\cong$ 83 $\cong$ 250

$\cong$ 125 $\cong$ 376 $\cong$ 188 $\cong$ 94

$\cong$ 47

142 $\cong$ 71 $\cong$ 224 $\cong$ 112

Now 112 comes in n = 2 series hence true.

n = 4 9.16 + 1 $\cong$ 3.9.16 + 4 $\cong$ 3.9.4. +1

$\cong$ 3.27.4 + 4 $\cong$ 81 + 1 = 82 $\cong$ 41 $\cong$ 124

$\cong$ 62 $\cong$ 31 110 $\cong$ 55

Now 55 comes in previous series.

n = 5 $9.2^5 + 1 \cong 3.9.2^5 + 4 \cong 27.2^3 + 1$

$\cong$ 27.3.23 + 4 $\cong$ 27.3.2. + 1 $\cong$ 3.81.2 +4

$\cong$ 3.81+ 2 $\cong$ 245 $\cong$ 736 $\cong$ 184 $\cong$ 92

$\cong$ 46 $\cong$ 23 $\cong$ 70 $\cong$ 35 $\cong$ 106 $\cong$ 53

$\cong$ 160 $\cong$ 80 $\cong$ 40

Now 40 comes in the series of n = 2

n = 6 $9.2^6 +1 \cong 3^3.2^6 + 4 \cong 3^3.2^6 + 4 \cong 3^3.2^4 + 1 \cong 3^4.2^4 + 4$

$\cong 3^4.\ 2^2 + 1 \cong 3^5.\ 2^2 + 4 \cong 3^5 +1 \cong 244$

$\cong$ 122 $\cong$ 66 $\cong$ 33 $\cong$ 100 $\cong$ 25 $\cong$ 76

$\cong$ 19 $\cong$ 58 $\cong$ 29 $\cong$ 88 $\cong$ 44 $\cong$ 22 $\cong$ 11

Which comes in n = 2 series

n = 7 9.2 + 1 $\cong$ 32+3. 2+ 1 $\cong$ 36 + 2 $\cong$ 731

$\cong$ 2194 $\cong$ 1097 $\cong$ 3292 $\cong$ 1646 $\cong$ 832 $\cong$ 2470

$\cong$ 1235 $\cong$ 3706 $\cong$1853 $\cong$5560 $\cong$2780

$\cong$ 1390 $\cong$695 $\cong$2086 $\cong$1043 $\cong$3130

$\cong$ 1565 $\cong$4696 $\cong$2348 $\cong$ 1174 $\cong$ 587

$\cong$1762 $\cong$ 881 $\cong$ 2644 $\cong$661 $\cong$ 1994

$\cong$997 $\cong$ 2992 $\cong$ 748 $\cong$ 187

$\cong$ 562 $\cong$ 281 $\cong$ 844 $\cong$ 211 $\cong$ 634

$\cong$317 $\cong$ 1552 $\cong$ 388 $\cong$ 97 $\cong$ 292

$\cong$73 $\cong$ 220 $\cong$ 55

Which comes in the series of n =3.

Henceforth for this seemingly easy looking conjecture seems to hold good.

Hence collatz conjecture is now a therorem.

It does have some useful implications too on our concept of numbers. The most important is that if any no. is subjected to the algorithm of Collatz problem then any of the no. s never repeat because if a no. appearing in the series comes again then it would be a perfect counter example because the chain would run from it to & fro.

Also a word for its various generalizations viz 2n+1 & n. r+ 1 rules, I guess they may be proved using similar methods (ie of mathematical inducation). Given the length of time it took to solve this one I am not going to try any one of them.

Do you have the guts ?

A mathematical challenge ....!

Author

Hemant Pandey
16/732, Indira Nagar, Lucknow.
Pin : 226 016
Uttar Pradesh, India
Tel : 91- 0522-2716255
Mob: 9336705954
9450097467
E-mail : ajay_euler@rediffmail.com
hemant_srk@rediffmail.com